Hitchin Generalized Geometry.

ref : Hitchin IMS lecture in 2010

§1. Basic M^n

From T to $T \oplus T^* \ni X+\xi$

$(X+\xi, X+\xi) = \iota_X \xi$ Signature (n,n).

Skew-adjoint endomorphism

$$\begin{pmatrix} A & \beta \\ B & -A^t \end{pmatrix}\begin{pmatrix} T \\ T^* \end{pmatrix} \to \begin{pmatrix} T \\ T^* \end{pmatrix}$$

$$(B(X_1+\xi_1), X_2+\xi_2) = (B(X_1), X_2) \quad = -(X_1, B(X_2))$$

$\Rightarrow$ $B : T \to T^*$ skew. i.e. $B \in \Lambda^2 T^*$

$\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}^2 = 0 \Rightarrow$ orthogonal auto. $X+\xi \mapsto X+\xi+\iota_X B$

$e^{\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}} = I + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$ B-field action.

- Lie bracket $\rightsquigarrow$ Courant bracket

$$[X+\xi, Y+\eta]$$
$$= [X,Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi)$$

Jacobi identity is not satisfied. i.e. not Lie alg.

Prop: $[\ ,\ ]$ is preserved by closed B-fields.

Pf. $[X+\xi+\iota_X B, Y+\eta+\iota_Y B]$

$$= [X+\xi, Y+\eta] + \underbrace{\mathcal{L}_X(\iota_Y B)} - \frac{1}{2} d(\iota_X \iota_Y B)$$
$$- \mathcal{L}_Y(\iota_X B) + \frac{1}{2} \underbrace{d(\iota_Y \iota_X B)}_{\underbrace{d \iota_Y \iota_X B}_{\mathcal{L}_Y(\iota_X B) - \iota_Y d(\iota_X B)}}$$

$$= [X+\xi, Y+\eta] + \iota_{[X,Y]} B \underbrace{+ \iota_Y \mathcal{L}_X B - \iota_Y d(\iota_X B)}_{\iota_Y \iota_X \cancel{dB}} \quad (\because B : \text{closed})$$

#

$\mathrm{Diff}\,M \ltimes \Omega^2_{cl}$ fund. gp. which we can transf. by.

$$X + \xi \in \Gamma(T \oplus T^*)$$

$$\mapsto X - d\xi \in \mathrm{Lie}(\Omega^2_{cl} \rtimes \mathrm{Diff}(M)) =: \mathfrak{g}$$

acts on section of $T \oplus T^*$

$$Y + \eta \curvearrowright \quad \mathcal{L}_X(Y+\eta) - \iota_X d\xi =: uv \qquad \begin{aligned} u &= X+\xi \\ v &= Y+\eta \end{aligned}$$

Indeed, Courant bracket $= \frac{1}{2}(uv - vu)$

$$\frac{1}{2}(uv + vu) = \frac{1}{2}(\mathcal{L}_X \eta - \iota_Y d\xi + \mathcal{L}_Y \xi - \iota_X d\eta)$$
$$= \frac{1}{2} d(\iota_X \eta + \iota_Y \xi)$$
$$= d(u, v).$$

Prop. $u(vw) = (uv)w + v(uw)$ i.e. deviation

Pf. Write $u = X + \xi$ & $\tilde{u} = X - d\xi$

$$u(vw) - v(uw) = \tilde{u}\tilde{v}(w) - \tilde{v}\tilde{u}(w) = [\tilde{u}, \tilde{v}](w)$$ ← Lie bracket in $\mathfrak{g}$

$$[\tilde{u}, \tilde{v}] = [X, Y] - ((\mathcal{L}_X d\eta - \mathcal{L}_Y d\xi))(w)$$

$uv = [X, Y] + \mathcal{L}_X \eta - \iota_X d\xi$, acts as $[X,Y] - d(\mathcal{L}_X\eta - \iota_Y d\xi)$

Note $d(\iota_Y d\xi) = \mathcal{L}_Y d\xi - \cancel{\iota_Y d^2\xi}$ #

Cor. For Courant bracket,

$$[[u,v],w] + \text{cyclic} = \frac{1}{3} d\left(\{([u,v],w) + \text{cyclic}\right).$$

Pf. LHS: $$\frac{1}{4}\begin{pmatrix} (\underbrace{uv}_{} - vu)w - w(uv - vu) \\ +(vw - wv)u - u(vw - wv) \\ +(wu - uw)v - v(wu - \underbrace{uw}_{\text{pair up}}) \end{pmatrix} \rightsquigarrow (-1)\times \text{ sum of right hand column.}$$

$\uparrow$ l $\qquad$ $\uparrow$ r

$$l + r = -r \quad \& \quad l - r = -3r = 3(l+r)$$

$$l + r = \frac{1}{3}(l - r)\frac{1}{4}\left\{\begin{matrix} \underbrace{(uv - vu)w + w(uv - vu)}_{} \\ + \\ + \end{matrix}\right\} \rightarrow 4\, d([u,v],w)$$

#

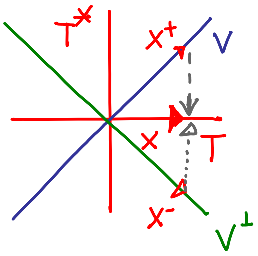
§2 Riemannian geometry.

$g : T \longrightarrow T^*$

$\frac{\partial}{\partial x_i} \mapsto g_{ij} dx_j$

$V :=$ graph of $g \underset{\text{subbdl}}{\subset} T \oplus T^*$

$V^\perp :=$ graph of $(-g)$.



Prop: If v is a section V
X vector field.

$$\nabla_X v := \underbrace{\pi_V}_{\text{proj. onto } V} [X^-, v]_{\text{Courant}}.$$

is a connection, preserves inner product induced on V.

reason: Properties of Courant bracket.

$$[u, fv] = f[u,v] + (Xf)v - (u,v)\, df.$$

$$X(v,w) = ([u,v] + d(u,v), w) + (v, [u,w] + d(u,w))$$

In fact, only need symmetric part of g to be pos.def.

$$\text{Tor}\,\nabla = d(\text{skew}(g)).$$

$$[\underbrace{\frac{\partial}{\partial x_i} - g_{i\ell}\, dx_\ell}_{X^- \in V^\perp}, \underbrace{\frac{\partial}{\partial x_j} + g_{jk}\, dx_k}_{\in V}]$$

$$= \frac{\partial g_{j\ell}}{\partial x_i} dx_\ell + \frac{\partial g_{i\ell}}{\partial x_j} dx_\ell - \frac{1}{2} \frac{\partial}{\partial x_\ell}(g_{ji} + g_{ij})\, dx_\ell$$

projection on V, (assume $g_{ij} = g_{ji}$)

$$\pi_V(dx_\ell) = \frac{1}{2}\left(dx_\ell + g^{\ell u} \frac{\partial}{\partial x_u}\right)$$

$$\rightsquigarrow \frac{1}{2} g^{\ell k}\left(\frac{\partial g_{j\ell}}{\partial x_i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x_\ell}\right)\left(g_{ui}\, dx_i + \frac{\partial}{\partial x_u}\right).$$

$\rightsquigarrow$ Christoffel symbol.

2. Bianchi IX metric

$$dt^2 + a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + a_3^2 \sigma_3^2$$

σ_i's: inv. forms on $SU(2)$

$a_i = a_i(t)$

$\mathcal{L}_{X_2} \sigma_1 = \sigma_3$.

$d\sigma_1 = \sigma_2 \wedge \sigma_3$

X_i : dual v.f. basis.

Courant bracket helpful for computations.

eg. $\nabla_{X_1}(a_2\sigma_2) = [X_1 - a_1^2\sigma_1 , a_2\sigma_2 + \frac{1}{a_2}X_2]$

$= \pi_V[-\frac{X_3}{a_2} - a_2\sigma_3 + \frac{a_1^2}{a_2}\sigma_3]$

$= \ -\frac{X_3 - a_3^2\sigma_3}{a_2} - a_2\sigma_3 + \frac{a_1^2}{a_2}\sigma_3) + \cdots$

$\Rightarrow \quad \nabla_{X_1}(a_2\sigma_2) = \frac{1}{2}(\frac{a_1^2}{a_2} - a_2 - \frac{a_3^2}{a_2})\sigma_3$.

§ 3 Spinors

$\varphi \in \Lambda^\bullet T^*$

$T\oplus T^* =: \mathcal{U} \ni X+\xi$

$O(n,n)$

$(X+\xi)\cdot\varphi := \iota_X\varphi + \xi\wedge\varphi$

$(X+\xi)^2\cdot\varphi = \cdots = (\iota_X\xi)\varphi = (X+\xi, X+\xi)\varphi$

⇝ Clifford alg. action

$Cl(\mathcal{U})$ $\qquad uv + vu = 2(u,v)$

$CL(\mathcal{U})$ $\qquad x_1\cdots x_n \to x_n\cdots x_1$

anti-auto $\qquad a \mapsto a^T$

bilinear form

S = Spin rep. = $\Lambda^\bullet T^*\otimes(\Lambda^n T^*)^{1/2}$

bilinear form $(\varphi_1,\varphi_2) \triangleq \sum(-1)^j(\varphi_1^{2j}\wedge\varphi_2^{n-2j} + \varphi_1^{2j+1}\wedge\varphi_2^{n-2j-1}) \in \mathbb{R}$

Mukai pairing.

Lie algebra of $SO(n,n) \subset CL(\mathcal{U}) \supset \mathcal{U}$

is $\{a : [a,\mathcal{U}] \subset \mathcal{U} \ \& \ a = -a^T\}$

$B \in \Lambda^2 T^*$

$B\wedge(\iota_X + \xi)\varphi - (\iota_X + \xi\wedge)B\wedge\varphi$

$= -(\iota_X B)\varphi - B\wedge\cancel{\iota_X\varphi} + \cancel{B\wedge\iota_X\varphi}$

⇒ B-field action on spinor = $\varphi \mapsto e^{-B\wedge}\varphi$.

$\Omega^2_{cl} \rtimes \mathrm{Diff}(M)$ action on spinor is

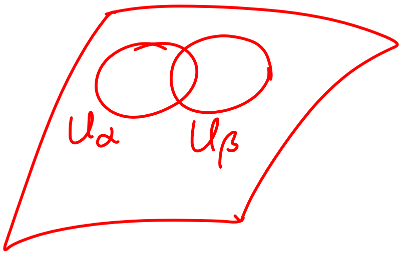
$$\mathcal{L}_X\varphi + d\xi\wedge\varphi$$

Cartan-like formula,

$$d(X+\xi)\cdot\varphi + (X+\xi)\cdot d\varphi = \underbrace{d\iota_X\varphi + \iota_X d\varphi}_{\mathcal{L}_X\varphi} + d\xi\wedge\varphi$$

§ Twisted structures

$T\oplus T^*$, $(\,,\,)$, $[\,,\,]$: preserved by closed B-field.



$$B_{\alpha\beta}\in\Omega^2_{cl}(U_\alpha\cap U_\beta)$$

$B_{\alpha\beta}+B_{\beta\gamma}+B_{\gamma\alpha}=0$ on $U_\alpha\cap U_\beta\cap U_\gamma$

$$(T\oplus T^*)|_{U_\alpha} \cong (T\oplus T^*)|_{U_\beta}$$

↑ identified by $X+\xi \mapsto X+\xi+\iota_X B_{\alpha\beta}$

⇝ give us a VB E, locally modelled on $T\oplus T^*$.

$$0\to T^*\to E\to T\to 0$$

E equips w/ $(\,,\,)$ & $[\,,\,]$.

Exact Courant algeboid.

exact seq. of sheaves, $0\to\Omega^2_{cl}\to\Omega^2\to\Omega^3_{cl}\to 0$

$$[B_{\alpha\beta}]\in H^1(\Omega^2_{cl}) = H^0(\Omega^3_{cl})/dH^0(\Omega^2) = H^3(M,\mathbb{R}).$$

$(E,(\,,\,))$ ⇝ Spinor bdl S

$\Lambda^\bullet T^*|_{U_\alpha}\cong\Lambda^\bullet T^*|_{U_\beta}$ via $\varphi\mapsto e^{-B_{\alpha\beta}}\varphi$

$e^{B_{\alpha\beta}}d(e^{-B_{\alpha\beta}}\varphi)=d\varphi$ (∵ $dB_{\alpha\beta}=0$)

$$d: C^\infty(S^{ev})\longrightarrow C^\infty(S^{odd})$$

⇝ twisted deRham cohomology

$$\begin{tikzcd} 0 \arrow[r] & T^* \arrow[r] & E \arrow[r] & T \arrow[r] \arrow[l, bend left, "\text{isotropic splitting}"] & 0 \end{tikzcd}$$

$\sim$ $F_\alpha \in \Omega^2(U_\alpha)$ s.t. $B_{\alpha\beta} = F_\beta - F_\alpha$

$H := dF_\alpha = dF_\beta$ $(\because dB_{\alpha\beta} = 0)$

$H \in \Omega^3(M)$.

Section of S, i.e. φ_α : forms on U_α s.t.
$\varphi_\alpha = e^{-B_{\alpha\beta}} \varphi_\beta$ on $U_\alpha \cap U_\beta$

$\Rightarrow$ $e^{-F_\alpha} \varphi_\alpha = e^{-F_\beta} \varphi_\beta$ ⇝ global form ψ

If $d\varphi_\alpha = 0$, then $d\psi = -(dF_\alpha)\psi$
i.e. $(d + H)\psi = 0$.

Remark: Such H's arises in gerbes.

Gerbes.

U(1)-gerbe	U(1)-bdl.
$g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\beta \to U(1)$	$g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$
Connection structure $A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = (g^{-1} dg)_{\alpha\beta\gamma}$	connect^n: $A_\beta - A_\alpha = g_{\alpha\beta}^{-1} dg_{\alpha\beta}$

$\Rightarrow$ $dA_{\alpha\beta} + dA_{\beta\gamma} + dA_{\gamma\alpha} = 0$ ⇝ 1-cocycle w/ values in closed 2-forms.

$0 \to T^* \to E \to T \to 0$ w/ $(\, , \,)$, $[\ ]$

Def^n: A generalized metric in E is a subbdl $V \subset E$ of rank n s.t. $(\, , \,)|_V > 0$.

Locally, V is a graph of $h_\alpha : T \to T^*$
$h_\alpha = g_\alpha + F_\alpha \in \mathrm{Sym}^2 T^* \oplus \Lambda^2 T^*$

On $U_\alpha \cap U_\beta$, $g_\alpha = g_\beta$ ⇝ global metric
$F_\beta - F_\alpha = B_{\alpha\beta}$.

$$\because \left.\begin{array}{l} (\ \ )|_V > 0 \\ (\ \ )|_{T^*} = 0 \end{array}\right\} \Rightarrow \begin{tikzcd} E \arrow[r] & T \arrow[r] & 0 \\ V \arrow[u, phantom, "\cup"] \arrow[ur, "\cong"'] & & \end{tikzcd}$$

⇝ $\nabla_X v = \pi_V [X^-, v]$.

$$\left[\frac{\partial}{\partial x_i} - g_{ik}\,dx_k + F_{ik}\,dx_k,\ \frac{\partial}{\partial x_j} + g_{jl}\,dx_l + F_{jl}\,dx_l\right]$$

$$= \text{Levi-Civita} + \frac{\partial F_{jl}}{\partial x_i}\,dx_l - \frac{\partial F_{ik}}{\partial x_j}\,dx_k - \frac{1}{2}\,d(F_{ji} - F_{ij})$$

$$= \left(\frac{\partial F_{jl}}{\partial x_i} - \frac{\partial F_{il}}{\partial x_j} - \frac{\partial F_{ji}}{\partial x_l}\right) dx_l$$

skew-torsion $\quad dF_\alpha = H \in \Omega^3(M)$

⇝ Riemannian metric w/ skew-torsion.

Example 1. Lie group G, w/ Bi-inv. metric $\quad \nabla_X Y = 0$ for left-inv. v.f.

⇝ flat w/ skew-torsion.

Example 2. Bismut connection. On a Hermitian mfd, ∃! connection w/ skew-torsion which preserves metric & cpx str..

$$H = d^c\omega = J\,d\omega \qquad (\because J\omega = \omega).$$

$dH = 0 \iff dd^c\omega = 0$. SKT metric
Strong Kähler w/ torsion

§ Generalized complex structure

$T \oplus T^*$, $(\ ,\ )$, $[\ ,\ ]$

Ordinary: $J : T \to T$ $\quad J^2 = -1$

$+i$ eigenspace in $T \otimes \mathbb{C} \rightsquigarrow T^{1,0} \ni X, Y$

integrability : $[X, Y] \in T^{1,0}$.

Def. A generalized cx. str. is

- $J : T \oplus T^* \circlearrowleft$, $J^2 = -1$
- $(Ju, v) = -(u, Jv)$
- $(+i)$- eigenspace $\subset (T \oplus T^*) \otimes \mathbb{C}$, call $E^{1,0}$

  sections closed under Courant bracket

Note: $[u, fv] = f[u,v] + (Xf)v - \underbrace{(u,v)}_{0} df$

$Ju = u \Rightarrow i(u,u) = -(Ju, v) = 0 \Longrightarrow$

$\Rightarrow$ tensorial.

Examples : 1. Cx. mfd. $E^{1,0} = \langle \frac{\partial}{\partial \bar{z}_i} , dz_i \rangle$

2\. Symplectic mfd. $E^{1,0} = \langle \frac{\partial}{\partial x} , \underbrace{-i\omega_{jk}\, dx_k}_{\text{B-field}} \rangle$.

$T \subset T \oplus T^*$

$\frac{\partial}{\partial x^j} \mapsto \frac{\partial}{\partial x^j} - i\omega_{jk}\, dx_k$

closed 2-form preserves $[\ ,\ ]$

3\. Holomorphic Poisson manifold

Recall. $\sigma \in \Gamma(\Lambda^2 T) \rightsquigarrow \sigma : T^* \to T$

$\sigma(df) = X_f$ Hamil. v.f.

$\{f, g\} := \sigma(df)(g) = \sigma(df, dg) = -\{g, f\}$

$\sigma(d\{f,g\}) \underset{\uparrow}{=} [X_f, X_g]$ Lie bracket

integrability condition.

$E^{1,0} = \langle \frac{\partial}{\partial \bar{z}_i} , dz_j - \sigma(dz_j) \rangle$

integ. ?

$$[dz_i - \sigma(dz_i), dz_j - \sigma(dz_j)]$$
$$= [\sigma(dz_i), \sigma(dz_j)] - d\{z_i, z_j\} + d\{z_j, z_i\} + \tfrac{1}{2} d(\{z_i, z_j\} - \{z_j, z_i\})$$
$$= \sigma\, d\{z_i, z_j\} - d\{z_i, z_j\}.$$

(or simply "B-field" transf. w/ σ).

$M^{2\mathbb{C}}$ complex surface $\Lambda^2 T = K^{-1}$
(integ. autom.) w/ $H^0(M, K^{-1}) \neq 0$

Note: $E^{1,0} \subset (T \oplus T^*)^c$ max. isotropic

$\sim$ pure spinor $\rightsquigarrow$ forms $\varphi \in \Lambda T^* \otimes \mathbb{C}$
(pure $\sim$ ann(φ) has max dim).

Eg. 1. $\varphi = dz_1 \wedge \cdots \wedge dz_n$ 2. $\varphi = e^{i\omega}$.
$\mathrm{Ann}(\varphi) = E^{1,0}$

Require $d\varphi = 0$ $\sim$ gen. CY.

Integ. $\Longleftrightarrow$ $d\varphi = \underset{\substack{\uparrow \\ \text{Clifford product}}}{\theta \cdot \varphi}$ $\exists\, \theta \in C^\infty((T \oplus T^*)^c)$

$$\begin{array}{c} uv \\ \shortparallel \\ \tilde{u}(v) \end{array} \qquad \begin{array}{ccc} u & \longmapsto & \tilde{u} \\ X + \xi & & X - d\xi \end{array} \in \mathrm{Lie}(\Omega^2_{cl} \rtimes \mathrm{Diff}(M))$$

L_u ~ Lie derivative action of $\tilde{u}$.

$$L_u \varphi = d(u \cdot \varphi) + u \cdot (d\varphi).$$

Assume $d\varphi = \theta \cdot \varphi$

$$u \cdot \varphi = 0 = v \cdot \varphi$$
$$0 = L_v(u \cdot \varphi) = (L_v u) \cdot \varphi + u \cdot \underbrace{L_v \varphi}_{\cancel{d(v \cdot \varphi)} + v \cdot d\varphi}$$
$$= (L_v u) \cdot \varphi + \underbrace{u \cdot v \cdot \theta \cdot \varphi}_{0} \Leftarrow (v \cdot \theta + \theta \cdot v = 2(v, \theta))$$

$$2[u, v]\varphi = (L_v u - L_u v) \cdot \varphi = 0$$ $\#$

Note. $df \in T^*\oplus T \qquad \ni J(df) =: X + \xi$

$X - d\xi$ is a symmetry of the gen. cx. str.

Symplectic $J(df) = X_f$ Hamiltonian v.f.

I-Cx. case $J(df) = I(df) \rightsquigarrow d(I\,df) \leftarrow$ closed (1,1) real form.

$\Omega^2_{cl} \rtimes \mathrm{Diff}$

§7. The $\bar\partial$-complex

$$(T\oplus T^*)^c = E^{1,0}\oplus E^{0,1}$$

$$\bar\partial_J f := (df)^{0,1}$$

1. cx mfd. $\bar\partial_J f = \bar\partial f$
2. sympl. mfd. $\bar\partial_J f = \frac{1}{2}(iX_f + df)$
3. holo. Poisson $\bar\partial_J f = \bar\partial f + \sigma(\partial f) - \bar\sigma(\bar\partial f)$

$$\bar\partial_J : C^\infty(M) \longrightarrow C^\infty(E^{0,1}) \qquad (E^{0,1})^* \simeq E^{1,0}$$

$$2\bar\partial\alpha(u,v) = X\alpha(v) - Y\alpha(u) + \alpha([u,v])$$

$$\begin{cases} u = X+\xi \\ v = Y+\eta \end{cases} \in E^{0,1}$$

extends

$$\rightsquigarrow \bar\partial_J : C^\infty(E^{0,1}) \longrightarrow C^\infty(\Lambda^2 E^{0,1})$$

$(\bar\partial_J)^2 = 0$? (Only work because of isotropic cond^{n}.) $\Rightarrow$ Jacobi identity holds for $C^\infty(E^{0,1})$)

$$\rightsquigarrow \bar\partial_J : C^\infty(\Lambda^p E^{0,1}) \longrightarrow C^\infty(\Lambda^{p+1} E^{0,1}), \quad \bar\partial_J^2 = 0$$

Ex. 1. cx str. $E^{0,1} = \overline{T}^* \oplus T$

$$\Lambda^p E^{0,1} = \bigoplus_{\ell+m=p} \Lambda^\ell \overline{T}^* \otimes \Lambda^m T$$

$$\bar\partial_J = \bar\partial : \bigoplus \Omega^{0,\ell}(\Lambda^m T) \longrightarrow \bigoplus \Omega^{0,\ell+1}(\Lambda^m T).$$

Twist. (on a honest cpx mfd.).

replace $T \oplus T^*$ by E.
(~ closed 3-form H

patch by $B_{\alpha\beta}$

$B^{1,1}_{\alpha\beta}$ - transf. (~ cpx. holo. bdl.)

⇝ Add $H^{1,2} : \Omega^{0,\ell}(\Lambda^m T) \longrightarrow \Omega^{0,\ell+2}(\Lambda^{m-1} T)$
to $\bar{\partial}_J$.

(elliptic complex.).

Note : 1st order deform of gen. cx. str. (at a cpx str. param. by $H^0(\Lambda^2 T) \oplus H^1(T) \oplus H^2(\mathcal{O})$.

Obstruction $\in$ $H^0(\Lambda^3 T) \oplus H^2(T) \oplus H^3(\mathcal{O})$.
$\oplus H^1(\Lambda^2 T)$ ......

§ 8 Generalized Kähler manifolds

Kähler ~ cpx + symp.
J_1 J_2 ← 2 gen. cx. str.

Def. A gen. Kähler mfd. consists of 2 gen. cx. str. J_1, J_2 satisfying

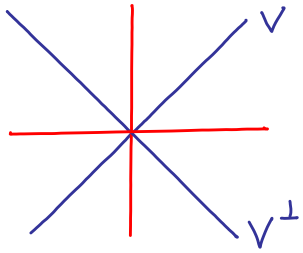
$$J_1 J_2 = J_2 J_1 \ \& \ (J_1 J_2 u, u) > 0 .$$

Theorem. M^{2n} Gen. Kähler ⇒
- metric g
- 2 integ. Hermitian cx. str. I^+, I^-
- metric conn. ∇^+, ∇^- w/ skew torsion $\pm H$
  s.t. $\nabla^+ I^+ = 0 = \nabla^- I^-$
- closed 2-form (equiv. up to B-field action).

(Gate-Hull-Rocek 1984).

$(J_1 J_2)^2 = 1 \quad \rightsquigarrow \quad (\pm 1)$ eigenspaces.

dim eigenspace $= 2n$ $\quad (\because \ (J_1 J_2 -,-) > 0)$.



$\rightsquigarrow$ metric w/ skew torsion

$$\begin{cases} J_i|_{V \simeq T} \rightsquigarrow I^+ \\ J_i|_{V^\perp \simeq T} \rightsquigarrow I^- \end{cases}$$

Theorem (Goto) Let M be a cpt. Kähler mfd w/ a holo. Poisson str. σ. Let $J_1(t)$ be the gen. cx. str. def$^{\underline{d}}$ by $t\sigma$, then for any small t $\exists$ analytic family of gen. Kähler str. $(J_1(t), J_2(t))$ where $J_2(0)$ = sympl. str. of the original Kähler structure.

Note. $[I^+, I^-] \in C^\infty(\Lambda^2 T)$ is real part of holo. Poisson.

Def. A generalized holo. bdl. on a gen. cx. mfd is a VB together with an operator

$$\bar{D} : C^\infty(V) \longrightarrow C^\infty(V \otimes E^{0,1})$$

such that $\quad \bar{D}(fs) = s \otimes \bar{\partial}_J f + f \bar{D} s \ , \quad \bar{D}^2 = 0$

Example : 1. Any gen. cx. str. has canonical bundle as gen. holo. bdl.

$K \subset (\Lambda^* T^*)^c$

$u \in E^{1,0} \iff u \cdot \varphi = 0 \quad$ for $\varphi \in (\Lambda^* T^*)^c$

fiber of $K = \mathbb{C} \cdot \varphi$

integrability : $\quad d\varphi = \theta \cdot \varphi \quad \exists ! \ \theta \in E^{0,1}$

$$\bar{D}(f\varphi) := (\bar{\partial}_J f + f\theta^{0,1}) \cdot \varphi$$ well-def$^{\underline{d}}$.

$$\bar{D}^2 = 0 \iff \bar{\partial}_J \theta = 0$$

$u, v \in E^{1,0}$, $\ (\bar{\partial}_J \alpha)(u,v) = X(\alpha, v) - Y(\alpha, u) + \alpha[u,v]$.

$$L_u \varphi = d(\cancel{u\cdot\varphi}) + u\cdot d\varphi = u\cdot\theta\cdot\varphi = -\theta\cancel{u\varphi} + 2(u,\theta)\varphi$$

$$L_v L_u \varphi = 2\,L_v(u,\theta)\varphi + 4(u,\theta)(v,\theta)\varphi$$

$$L_u L_v \varphi = 2\,L_u(v,\theta)\varphi + 4(v,\theta)(u,\theta)\varphi$$

$$\overset{-}{\rightsquigarrow} \; = \frac{1}{2}[u,v]\,\theta\cdot\varphi = 2\,\frac{1}{2}([u,v],\theta)\varphi .$$

#

Eg. 2. J = ordinary cx. structure.

$$\bar{D} : V \longrightarrow V\otimes(T^{0,1*}\oplus T^{1,0})$$

$$\bar{D}(fs) = s\otimes(\underbrace{\bar\partial_J f}_{\text{holo.}}, 0) + f\,\bar{D}s$$

$$\bar{D}s = (\underbrace{\bar\partial_A s}_{\text{usual holo. str. for } V}, \phi s) \qquad \phi\in C^\infty(\mathrm{End}\,V\otimes T^{1,0})$$

$$0 = \bar{D}^2 s = (\underset{\substack{\downarrow\\ \text{holo. str. on } V}}{\bar\partial_A^2 s}, \underset{\phi \text{ is holo}}{(\bar\partial_A\phi)s}, \underset{\phi\wedge\phi = 0 \in H^0(\mathrm{End}V\otimes\Lambda^2 T)}{\phi\wedge\phi\, s})$$

§ Generalized Complex Submanifolds.